Chapter 2

Optimal Control Theory

2.1. Optimal Launch of a Satellite

We start this chapter with what might be called the fundamental trajectory optimization problem: launching a satellite into orbit. Before setting up the mathematical model for this problem, let us first discuss in words what the problem entails.

It is very expensive to launch a satellite into orbit about the Earth because of the tremendously high speed required to achieve a low circular orbit. The lowest energy orbit must be high enough above the sensible atmosphere to avoid immediate dissipation of its energy and subsequent orbit decay and reentry. The lowest "sustainable" orbit is about 160 km (100 miles) above the Earth's surface. At this altitude the satellite must travel at 7.9 km/s (5 miles/s) to stay in circular orbit. In its final flight in 2011, the cost of reaching circular orbit with the United States Space Shuttle was about \$22,000/kg (\$10,000/lb). So if we can save a few kilograms on the way into orbit, we can save a lot of money.

Let's consider what our choices are in launching a satellite into circular orbit. To make things easier, we will cheat a little and ignore atmospheric drag during launch. Let us also assume that the launch vehicle has a predetermined amount of propellant and that it burns the propellant at a constant rate. Then, if the specific impulse is constant, the thrust of the engine, F, is constant. To further simplify, we assume the rocket has a single stage. (There is, of course, a great advantage to staging the engines, but that is a parameter optimization problem as discussed in the exercises in Chap. 1.)

Let us also assume: the Earth is not rotating (so we do not get a boost from the rotation), the circular orbit plane contains the launch site (so there is no out-of-plane dog-leg maneuver), the Earth is flat (we tip our hats to the Flat Earth Society), and the acceleration due to gravity is a constant equal to standard free fall (i.e. $g = 9.80665\ \mathrm{m/s^2}$).

One might well ask, "What is there left to do?" It turns out that we have one control variable. In Fig. 2.1, we illustrate our flat-Earth problem of launching a satellite into circular orbit. The orbital plane is in the xy plane, and the altitude we wish to achieve corresponds to a radial distance from the center of the Earth, r_c. The one control variable we have is the steering angle of the thrust, α.

Consider what happens when a rocket is launched into orbit. First, the rocket lifts off the pad and rises vertically, which corresponds to a steering angle of 90°. Then, as

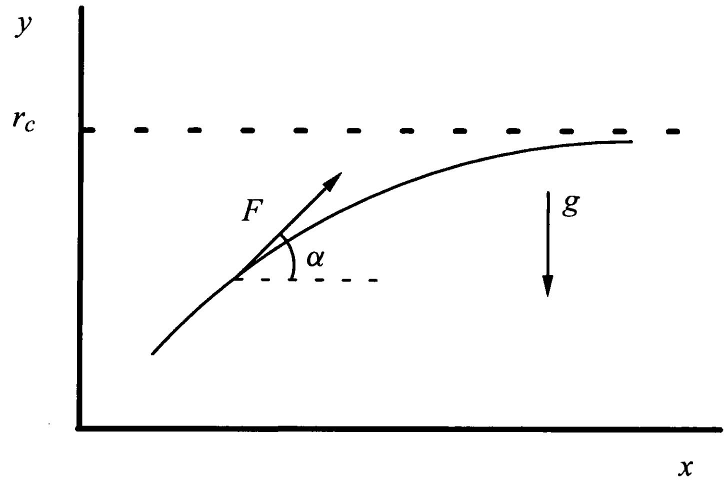

Figure 2.1. Problem of finding the steering law, $\alpha(t)$, to maximize satellite mass.

the rocket achieves higher altitudes, it begins to pitch over so that α is less than 90°. Eventually, the rocket is traveling with α near zero (or even negative to cancel upward momentum).

Here we note that actual launches from the Earth have to contend with air drag so that α remains near 90° longer than would be necessary if there were no atmosphere. In launching astronauts from the Moon, the steering angle α approached zero (and below) much more rapidly, because there was no atmosphere to clear (just high lunar mountains).

The time function $\alpha(t)$ could follow an infinite number of profiles that would achieve circular orbit. For every steering profile, there is a corresponding path of the rocket given by $x(t)$ and $y(t)$.

Our problem is to find $\alpha(t)$ such that the maximum payload is delivered into a prescribed circular orbit. We are maximizing final mass, which for a constant burn rate means we are minimizing the time to get into orbit. If we minimize the time for a given burn rate, we minimize the propellant burned. Our infinite-dimensional problem is: find the steering law, $\alpha(t)$, which is a function of time (an infinite number of points) to minimize the final time to orbit, t_f.

Example 2.1 Launch into circular orbit.

We assume that at the initial time, t_o, we know the position and velocity components, x_o, y_o, v_{xo}, and v_{yo}. However, we do not know the final time, t_f, nor the final value, x_f, but we do know the final velocity components $v_{xf} = v_c$ and $v_{yf} = 0$, which are specified for the circular orbit at the known altitude, $y_f = r_c - r_{\text{Earth}}$, where r_{Earth} is the radius of the Earth.

The mass of the rocket is

$$m(t) = m_o + \dot{m}_o(t - t_o) \tag{2.1}$$

where $\dot{m}_o$ is a negative constant.

For our "flat-Earth" problem in Fig. 2.1, the governing differential equations are

$$\dot{x} = v_x \tag{2.2a}$$

$$\dot{y} = v_y \tag{2.2b}$$

$$\dot{v}_x = \frac{F}{m}\cos\alpha \tag{2.2c}$$

$$\dot{v}_y = \frac{F}{m}\sin\alpha - g \tag{2.2d}$$

where Eqs. (2.2c) and (2.2d) are from Newton's second law. In our problem we must find $\alpha(t)$ to minimize the time.

Next, we introduce standard nomenclature from the literature. We write state variables as:

$$x_1 = x \tag{2.3a}$$

$$x_2 = y \tag{2.3b}$$

$$x_3 = v_x \tag{2.3c}$$

$$x_4 = v_y \tag{2.3d}$$

For our control variable we use:

$$u = \alpha \tag{2.4}$$

Using vector notation (indicated by bold) we can write our differential equations, Eqs. (2.2), and initial conditions as follows:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, u) \tag{2.5a}$$

$$\boldsymbol{x}(t_o) = \boldsymbol{x}_o \tag{2.5b}$$

where $\boldsymbol{x}$ and $\boldsymbol{f}$ are 4-vectors in this problem (and n-vectors in general).

For our final boundary conditions, we have the components

$$\Psi_1 \equiv x_2(t_f) - r_c + r_{\text{Earth}} = 0 \tag{2.6a}$$

$$\Psi_2 \equiv x_3(t_f) - v_c = 0 \tag{2.6b}$$

$$\Psi_3 \equiv x_4(t_f) = 0 \tag{2.6c}$$

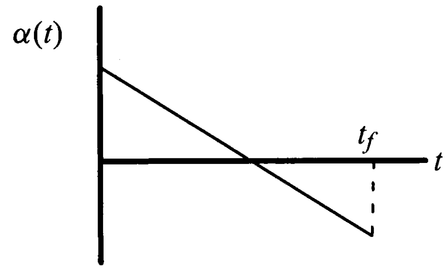

Figure 2.2. A possible steering law for minimum time to orbit.

or

$$\boldsymbol{\Psi}(t_f, \boldsymbol{x}_f) = \mathbf{0} \tag{2.7}$$

where $\boldsymbol{\Psi}$ is a 3-vector (q-vector in general).

For the cost functional (or performance index) to be minimized we have:

$$J = t_f \tag{2.8}$$

which minimizes the time (and maximizes the payload). Figure 2.2 shows a representative plot of the optimal steering law. Sometimes the steering law can look that simple. In Chap. 3 we develop the theory and in Chap. 4 we apply the theory to find such control laws.

2.2. General Form of the Problem

The general form for the optimal control problem is expressed as:

Minimize:

$$J = \phi(t_f, \boldsymbol{x}_f) + \int_{t_o}^{t_f} L(t, \boldsymbol{x}, \boldsymbol{u})dt \tag{2.9}$$

subject to:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{u}) \tag{2.10a}$$

$$\boldsymbol{x}(t_o) = \boldsymbol{x}_o \tag{2.10b}$$

$$\boldsymbol{u} \in U \tag{2.10c}$$

$$\boldsymbol{\Psi}(t_f, \boldsymbol{x}_f) = \mathbf{0} \tag{2.10d}$$

with a possible state variable inequality constraint:

$$S(\boldsymbol{x}) \geq 0 \tag{2.11}$$

Here U is the set of admissible controls, which we will discuss later, and $S(\boldsymbol{x}) \geq 0$ is an inequality constraint which, for example, may require a spacecraft or aircraft to never fly below the surface of the Earth.

At this point it is useful to formalize our notation according to standard notation. Lower case letters in bold denote a vector

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{2.12}$$

A superscript T indicates the transpose of a vector or matrix:

$$\boldsymbol{x}^T\boldsymbol{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n \tag{2.13}$$

so that $\boldsymbol{x}^T\boldsymbol{y}$ represents the scalar product of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.

The gradient of a scalar function, J, is defined to be a row vector:

$$\begin{aligned} \nabla J = J_{\boldsymbol{x}} = \frac{\partial J}{\partial \boldsymbol{x}} &= \begin{bmatrix} \frac{\partial J}{\partial x_1} & \frac{\partial J}{\partial x_2} & \cdots & \frac{\partial J}{\partial x_n} \end{bmatrix} \\ &= \begin{bmatrix} J_{x_1} & J_{x_2} & \cdots & J_{x_n} \end{bmatrix} \end{aligned} \tag{2.14}$$

The second derivative of a scalar function with respect to vector arguments is:

$$\begin{aligned} J_{\boldsymbol{xy}} &= \frac{\partial}{\partial \boldsymbol{y}}(J_{\boldsymbol{x}}^T) \\ &= \begin{bmatrix} \frac{\partial^2 J}{\partial x_1 \partial y_1} & \frac{\partial^2 J}{\partial x_1 \partial y_2} & \cdots & \frac{\partial^2 J}{\partial x_1 \partial y_m} \\ \vdots & & & \\ \frac{\partial^2 J}{\partial x_n \partial y_1} & & \cdots & \frac{\partial^2 J}{\partial x_n \partial y_m} \end{bmatrix} \end{aligned} \tag{2.15}$$

Example 2.2 The brachistochrone problem.

In 1696, Johann Bernoulli posed and solved the brachistochrone problem (see Bell [1965]). He originally created the problem as a challenge to his brother, Jacob, who also solved it. Figure 2.3 illustrates the problem. The xy plane is vertical, in a uniform gravity field, where the x axis points downward in a uniform gravity field. A particle of mass m is placed (motionless) on a frictionless track that connects the origin to a given point (x_f, y_f). The shape of the track, $y = y(x)$, must be found such that the particle takes the shortest time to travel from the origin to (x_f, y_f). A straight line (i.e., a ramp) does not provide the quickest trajectory.

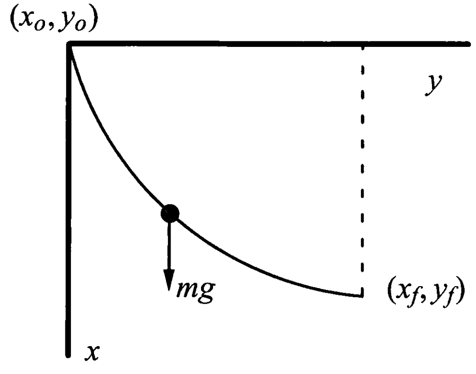

Figure 2.3. The brachistochrone problem: find $y = y(x)$ to minimize the time for the particle to slide in the vertical plane from $(x_o,\ y_o)$ to $(x_f,\ y_f)$. Note that the straight line solution is not the minimum-time path

In fact, the fastest trajectory starts with a steeper slope than the ramp to gain speed early and the extra speed more than makes up for the longer path. Since the initial speed of the particle is zero and it falls in a uniform gravity field, its velocity is known from the conservation of total mechanical energy:

$$v = \sqrt{2gx} \tag{2.16}$$

The infinitesimal distance, ds along the track can be given as

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (dy/dx)^2}\, dx = \sqrt{1 + y'^2}\, dx \tag{2.17}$$

Thus, the time to travel from $x = 0$ to $x = x_f$ is:

$$t = \int_0^{x_f} \frac{\sqrt{1 + y'^2}}{\sqrt{2gx}}\, dx \tag{2.18}$$

The problem is to find the path, $y = y(x)$, to minimize the time given by Eq. (2.18). This problem of quickest descent baffled the mathematicians of Europe for 6 months. Eventually several correct solutions were sent to Johann Bernoulli including an anonymous one from Isaac Newton, who solved the problem in one day. Upon receiving Newton's solution, Bernoulli exclaimed, "tanquam ex ungue leonem," loosely translated by Bell [1965] as "I recognize the lion by his paw print!"

It turns out that the solution is a cycloid, but much more important results ensued. The general problem could be written as

$$J = \int_{x_o}^{x_f} F[y(x), y'(x), x]dx \tag{2.19}$$

where J is a scalar, F is known, but $y(x)$ is unknown. The general problem is to find $y(x)$ that makes J stationary (such that small changes in $y(x)$ make no change in J). This was a new type of problem. How does one go about finding $y(x)$? How can we solve for an entire function?

It was discovered that $y(x)$ must satisfy the Euler-Lagrange equation:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad . \tag{2.20}$$

Equation (2.20) is a differential equation, and we know how to solve this type of problem. Solving a differential equation is tantamount to finding a function, which can be very difficult if the equation is nonlinear. [The derivation of Eq. (2.20) is given in Sect. 3.2.]

The brachistochrone problem led to a more general optimization problem and to a field of mathematics called the calculus of variations. The problem of launching a satellite into orbit is closely related to the brachistochrone problem. For the launch problem we must find the trajectory $x(t)$, $y(t)$ which gives the shortest time to orbit. The launch problem is an example of an optimal control problem in which we must find the steering law, $\alpha(t)$, to minimize the time.

Johann Bernoulli's discovery also led to the realization that dynamical motion obeys Lagrange's equations. That is, the functional

$$J = \int_{t_0}^{t_f} L\, dt \tag{2.21}$$

has a stationary value. The integrand, L, in Eq. (2.21) is referred to as the *Lagrangian*, which is equal to the difference between the kinetic and potential energies of the system.

This idea is known as Hamilton's principle which indicates that nature obeys an optimization principle. Such concepts occupied the minds of mathematicians and philosophers for several centuries. (For more information see the enthusiastic presentation given by Lanczos [1986]).

2.3. The Problems of Bolza, Lagrange, and Mayer

Throughout this text, we will be mainly concerned with the Problem of Bolza:

Minimize:

$$J = \phi(t_f, \boldsymbol{x}_f) + \int_{t_o}^{t_f} L(t, \boldsymbol{x}, \boldsymbol{u})dt \tag{2.22}$$

subject to:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{u}) \quad \text{System or Process Equations} \tag{2.23a}$$

$$\boldsymbol{x}(t_o) = \boldsymbol{x}_o \quad \text{Initial Conditions (I.C.s)} \tag{2.23b}$$

$$\boldsymbol{\Psi}(t_f, \boldsymbol{x}_f) = \boldsymbol{0} \quad \text{Terminal Constraints} \tag{2.23c}$$

where,

$\boldsymbol{x}$ is an n-vector

$\boldsymbol{u}$ is an m-vector

$\boldsymbol{\Psi}$ is a $q \leq n$ vector

and J is the scalar cost to be minimized. (See Bolza [1961], Bryson and Ho [1975], and Hull [2003].) The scalar J is called a functional because it maps functions [the path $\boldsymbol{x}(t)$ and control $\boldsymbol{u}(t)$] into a single number. The functional is also called the cost index, the performance index, or sometimes the cost function (though cost functional is more precise). As we have seen, typical examples for J are the propellant used to launch a spacecraft into orbit or the time for a particle to travel between points. Besides the Problem of Bolza, there are two other forms that appear in the literature.

In the Problem of Lagrange we have:

Minimize:

$$J = \int_{t_o}^{t_f} L(t, \boldsymbol{x}, \boldsymbol{u})dt \tag{2.24}$$

subject to the same conditions as the Problem of Bolza.

Equation (2.24) consists of what is sometimes referred to as the "path cost" which is the same form as we saw in the brachistochrone problem [Eq. (2.19)]. We will find this integral form useful when we explore the effect of variations of the path $\boldsymbol{x}$ and the control $\boldsymbol{u}$ from the optimal, which will lead us to the Euler-Lagrange equation, Eq. (2.20). In Eq. (2.24) the integrand, L, is called the Lagrangian. (We note for those familiar with Lagrangian dynamics that the integrand of Eq. (2.24) is more general and may have nothing to do with dynamics.)

In the Problem of Mayer we have:

Minimize:

$$J = \phi(t_f, \boldsymbol{x}_f) \tag{2.25}$$

subject to the same conditions as the Problem of Bolza.

Equation (2.25) is sometimes called the "terminal cost." It has the same form as Eq. (2.8) where we considered the problem of minimizing the final time to launch a satellite into orbit.

Example 2.3 Interchangeability of forms.

The three forms (Bolza, Lagrange, and Mayer) are equally general and interchangeable. (See Bliss [1968], Hull [2003], and Vagners [1983].) As a trivial example, consider:

$$\text{Mayer: Min. } J = t_f \tag{2.26}$$

$$\text{Lagrange: Min. } J = \int_0^{t_f} dt \tag{2.27}$$

$$\text{Bolza: Min. } J = \frac{1}{2}t_f + \frac{1}{2}\int_0^{t_f} dt \tag{2.28}$$

2.3.1. Transformation from Lagrange to Mayer

Let us consider in general how to transform the Problem of Lagrange into the Problem of Mayer. In the Problem of Lagrange we have:

Minimize:

$$J = \int_{t_o}^{t_f} L(t, \boldsymbol{x}, \boldsymbol{u})dt \tag{2.29}$$

Let us define a new variable with zero initial condition:

$$\dot{x}_{n+1} = L(t, \boldsymbol{x}, \boldsymbol{u}) \tag{2.30a}$$

$$x_{n+1}(t_o) = 0 \tag{2.30b}$$

Since $x_{n+1}(t) = \int_{t_o}^{t} L(t, \boldsymbol{x}, \boldsymbol{u})dt + x_{n+1}(t_o)$, the problem becomes:

Minimize:

$$J = x_{n+1}(t_f) \tag{2.31}$$

which is in Mayer form.

2.3.2. Transformation from Mayer to Lagrange

Next we show how to transform a Mayer problem into a Lagrange problem. For the Problem of Mayer we have:

Minimize:

$$J = \phi(t_f, \boldsymbol{x}_f) \tag{2.32}$$

To transform Eq. (2.32) into a Lagrange problem we write:

Minimize:

$$J = \int_{t_o}^{t_f} \frac{d\phi(t, \boldsymbol{x})}{dt} dt \tag{2.33a}$$

subject to:

$$\phi(t_o, \boldsymbol{x}_o) = 0 \tag{2.33b}$$

2.4. A Provocative Example Regarding Admissible Functions

Example 2.4 Admissible functions.

In this example we demonstrate how the class of functions being considered (for the state and the control) can affect the optimal solution we obtain.

Minimize:

$$J = \int_0^3 x^2 dt \tag{2.34}$$

subject to:

$$\dot{x} = u \tag{2.35a}$$

$$x(0) = 1 \tag{2.35b}$$

$$\Psi(t_f, x_f) = x(3) - 1 = 0 \tag{2.35c}$$

where x and u are scalars.

Problem: Determine $u(t) \in$ P.C. [0,3] which causes $x(t) \in C^0$ [0,3].

That is, our problem is to find a scalar control, $u(t)$, that is piecewise continuous over the closed time interval from $t_0 = 0$ to $t_f = 3$ such that $x(t)$ is continuous over the same time interval and such that Eq. (2.34) is minimized. The problem turns out to be a trick question, because there is no such control.

The point of this example is to emphasize the importance of the class of functions we are considering. The class of functions that we allow can have a dramatic effect on the solutions we obtain. For example, in Chap. 1 where we considered two impulsive ΔV maneuvers for orbit transfer between two circular coplanar orbits, we had the Hohmann transfer to minimize the total ΔV. However, when we allowed three impulses, we found it sometimes provided a lower total ΔV. Thus the type of control that we permit (i.e., what we call an admissible control) can affect the optimal solution.

A nice aspect of the problem we are now considering is that we don't need any theorems from optimal control to understand the problem. We can deduce all of our results by inspection. Upon examining the cost functional of Eq. (2.34), we see that

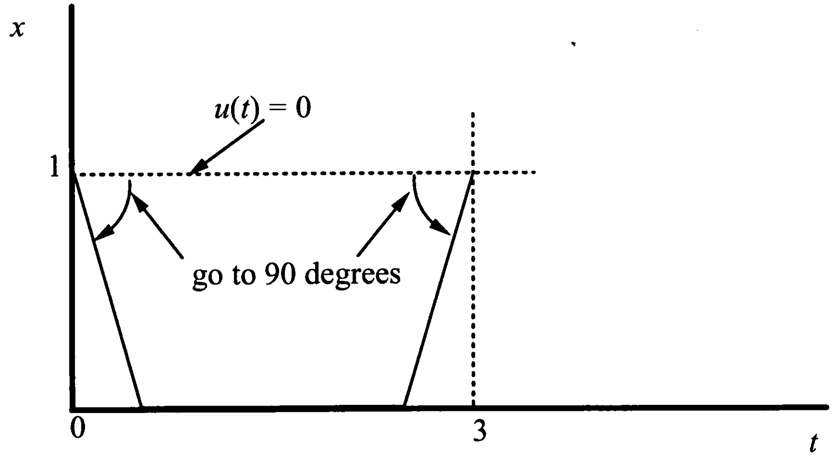

Figure 2.4. A provocative example regarding admissible functions. As the slope of x approaches vertical, the cost, Eq. (2.34), decreases until the slope is vertical when the control consists of Dirac delta functions

the value of x must be as close to zero as possible over the time interval. However, from the initial and final conditions [Eqs. (2.35b) and (2.35c)] the value of x must be unity at $t = 0$ and $t_f = 3$.

Figure 2.4 illustrates the solution for $u(t) \equiv 0$ where $x(t) \equiv 1$ is indicated by a dotted line. Clearly this is not the optimal. We note in the figure that if the value of x slopes downward near $t = 0$ and upward near $t_f = 3$ then the value for J will be small. If we consider the extreme case then x will discontinuously drop to zero at $t_0 = 0$ and will discontinuously jump to unity at $t_f = 3$. This corresponds to the 90° case indicated in Fig. 2.4.

If we allow these discontinuities then $x(t)$ will not be a continuous function as we originally assumed. Furthermore, $u(t)$ will not be a piecewise continuous function, but will instead appear as illustrated in Fig. 2.5, where we represent a Dirac delta function by the vertical arrows. On the other hand, if we allow the slope of $x(t)$ to approach 90° at the initial and final times, but to not actually reach 90°, then $x(t)$ will be continuous and $u(t)$ will be piecewise continuous. However, no optimal solution exists within these classes of functions because for any given slope near 90° there is always a steeper slope which gives a lower cost. When no unique solution exists, the problem has no optimal solution for the given class of functions.

We now provide more formal definitions of some classes of functions.

Piecewise Continuous: $x(t) \in$ P.C. $[t_o, t_f]$ if $x(t)$ is continuous at each open subinterval and if $\dot{x}$ has finite limits at the ends of the intervals. (See Fig. 2.6.)

m Continuously Differentiable: $x(t) \in C^m [t_o, t_f]$ if all derivatives of $x(t)$ of order $\leq m$ exist and are continuous. We note that $C^m \in C^{m-1}$ and that $C^1 \in C^0 \in$ P.C.

In Fig. 2.7, we show an example where x is continuous ($x \in C^0$) and $\dot{x}$ is piecewise continuous ($\dot{x} \in$ P.C.). The point on x with the jump is called a *corner*; the class is also referred to as *piecewise smooth*.

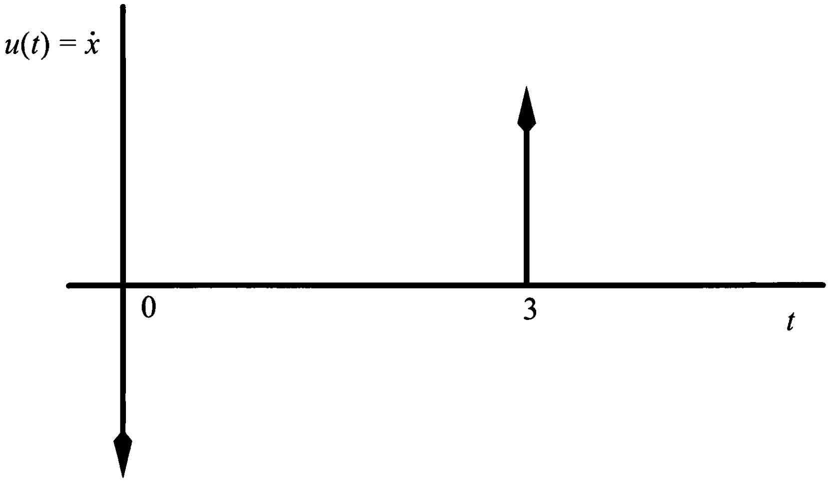

Figure 2.5. **The control for Fig. 2.4, $u(t) = \dot{x}(t)$. The *downward* and *upward arrows* represent Dirac delta functions, $-\delta(t)$ and $\delta(t-3)$, which are considered inadmissible.**

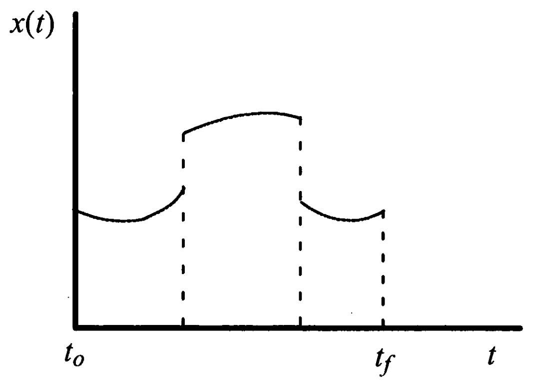

Figure 2.6. **An example of a piecewise continuous function**

The Dirac delta (or unit impulse) *function* has the properties:

$$\delta(t) = 0 \;\; \forall \, t \neq 0 \tag{2.36}$$

that is, the delta function is zero for all time not equal to zero and

$$\int_{-\infty}^{\infty} \delta(t) dt = 1 \tag{2.37}$$

This function has no definitive value at $t = 0$. According to Kaplan [1962], "...no ordinary function can have the properties mentioned. The situation is similar to that encountered in algebra: The equation $x^2 = -1$ can be satisfied by no real number.

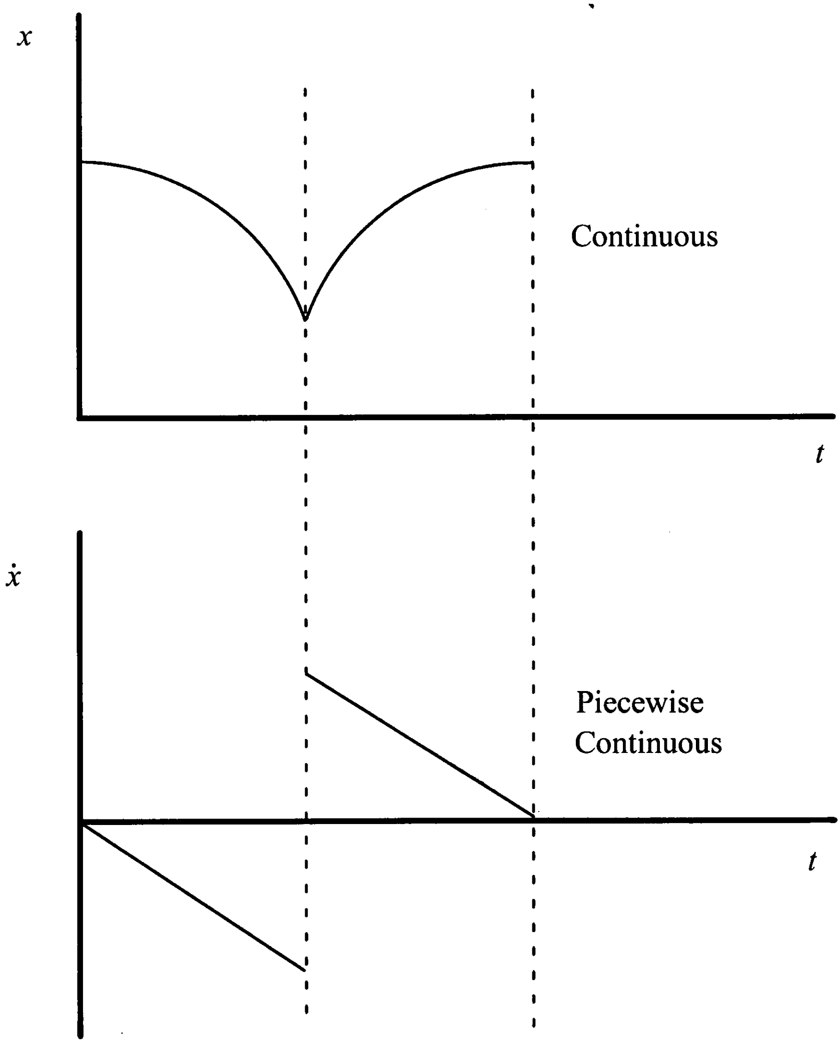

Figure 2.7. An example of a continuous function, $x \in C^0$, and its derivative which is piecewise continuous, i.e. $\dot{x} \in$ P.C. Continuous functions that have corners (discontinuous jumps in slope) are also called *piecewise smooth*

Hence we invent an 'imaginary number i' which has this property: $i^2 = -1$. In the same way we invent an 'imaginary' or 'ideal' function $\delta(t)$ to have the properties above." Kaplan goes on to develop a class of generalized functions that include δ, δ', δ'' etc. There is no mechanical way to generate such generalized functions and so they are not considered admissible.

Now, let us reconsider our problem given in Eqs. (2.34) and (2.35) in which we impose a new condition: $|u| \leq 1$. Then, the solution shown in Fig. 2.8 is obtained where $x(t) \in C^0[0, 3]$ and $u(t) \in P.C.[0, 3]$, which is admissible. We have achieved our goal by simply adding a constraint on the control.

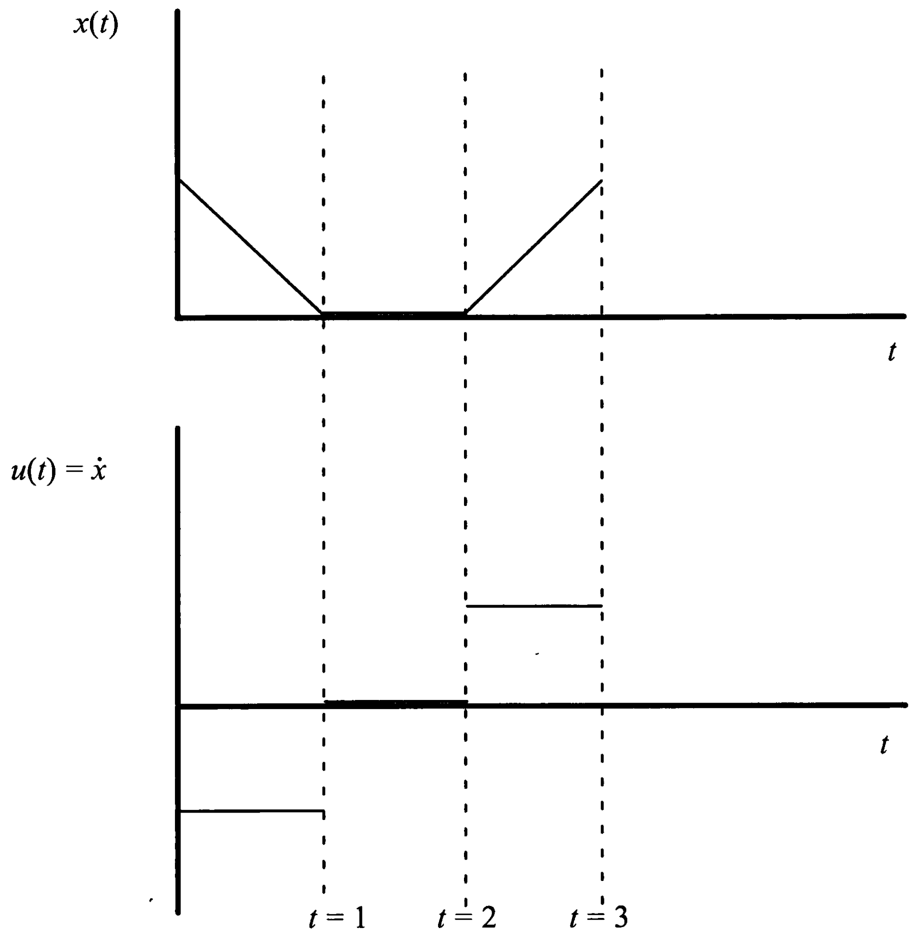

Figure 2.8. By limiting the magnitude of *u*: |*u*| ≤ 1, an admissible control to minimize Eq. (2.34) can be found

Next we consider yet another version of our problem. Suppose for some physical or mechanical reason that P.C. functions are inadmissible for the control. We assume in this case that $u \in C^0$ is admissible.

Let us define the following state and control variables:

$$x_1 = x \tag{2.38a}$$

$$x_2 = u \tag{2.38b}$$

$$\tilde{u} = \dot{u} \tag{2.38c}$$

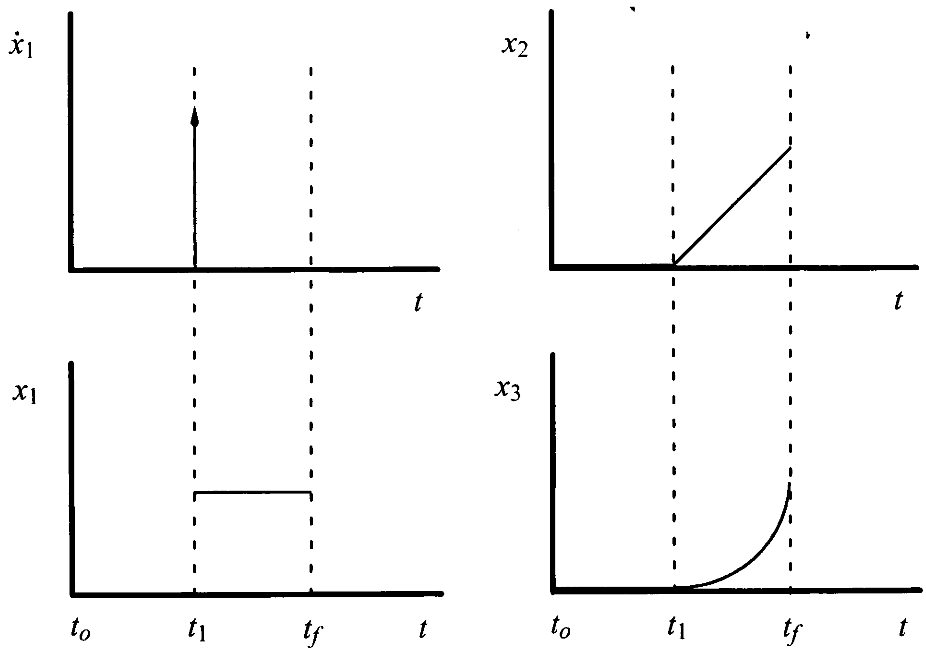

Figure 2.9. Example of classes of functions: $\delta(t - t_1)$, P.C., C^0, and C^1 over $[t_o, t_f]$.

and recalling from Eq. (2.35) that $\dot{x} = u$ we obtain

$$\dot{x}_1 = x_2 \tag{2.39a}$$

$$\dot{x}_2 = \tilde{u} \tag{2.39b}$$

By constraining $|\tilde{u}| \leq k$ (the slope of u), and $|x_2| \leq 1$ (the slope of x_1), we keep $u \in C^0[0, 3]$.

Example 2.5 Classes of functions.

In the following example let

$$\dot{x}_1 = \delta(t - t_1) \quad \text{for } t_o \leq t \leq t_f \tag{2.40a}$$

$$\dot{x}_2 = f(x_1) \tag{2.40b}$$

$$\dot{x}_3 = f(x_2) \tag{2.40c}$$

where $f \in C^\infty[t_o, t_f]$, which means that f is continuous (infinitely continuously differentiable) for all partial derivatives. What class is each of x_1, x_2, and x_3?

We can most easily visualize the solution by letting $f(x) = x$ in Eq. (2.40). Then Fig. 2.9 shows plots of $\dot{x}_1, x_1, x_2$, and x_3 and we have the result that

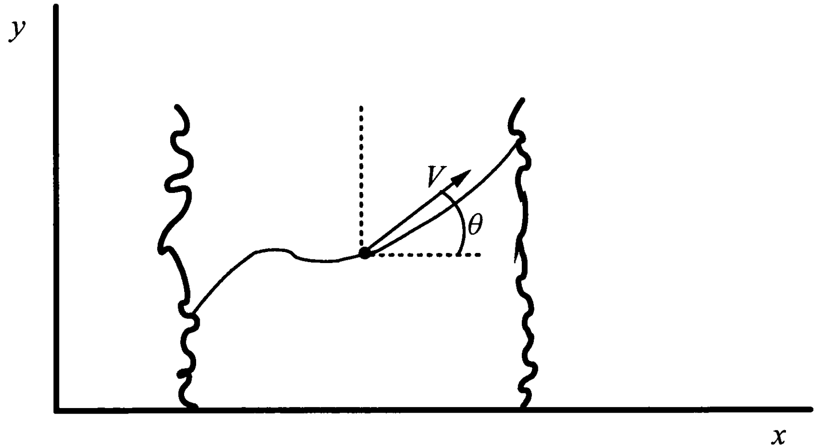

Figure 2.10. Zermelo's problem: a boat crosses a river in minimum time using θ as a control while V is a constant magnitude.

$$x_1(t) \in \text{P.C.}[t_o, t_f] \tag{2.41a}$$

$$x_2(t) \in C^0[t_o, t_f] \tag{2.41b}$$

$$x_3(t) \in C^1[t_o, t_f] \tag{2.41c}$$

Reinstating the function f into Eqs. (2.40) does not alter the conclusions of Eqs. (2.41).

Example 2.6 Zermelo's problem.

This problem involves minimizing the time required for a boat to cross a river, as illustrated in Fig. 2.10. In this example we assume the velocity V, is constant and that the control is the steering angle, θ.

Minimize:

$$J = t_f \tag{2.42}$$

subject to:

$$\dot{x} = V\cos\theta \tag{2.43a}$$

$$\dot{y} = V\sin\theta \tag{2.43b}$$

with B.C.s:

$$x(0) = x_o \tag{2.44a}$$

$$y(0) = y_o \tag{2.44b}$$

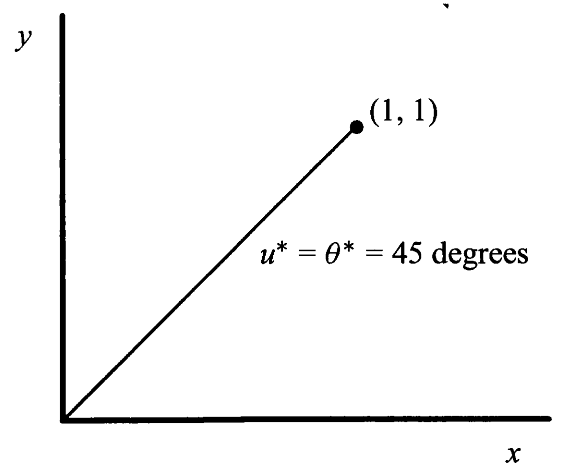

Figure 2.11. Solution to simplified Zermelo's problem: $\theta^* = 45°$.

$$x(t_f) = x_f \tag{2.44c}$$

$$y(t_f) = y_f \tag{2.44d}$$

We can simplify the problem further by assuming

$$x_0 = y_0 = 0 \tag{2.45a}$$

$$x_f = y_f = 1 \tag{2.45b}$$

This version of Zermelo's problem gives the optimal solution $\theta^* = 45°$, as shown in Fig. 2.11.

A more difficult problem considers currents, p and q, in the river, where the state equations are

$$\dot{x} = V\cos\theta + p \tag{2.46a}$$

$$\dot{y} = V\sin\theta + q \tag{2.46b}$$

and where p and q may be functions of time and of the state variables. We will return to examples of Zermelo's problem later in the text. We note in passing that an aerospace application of Zermelo's problem is the problem of an aircraft flying in a crosswind.

Example 2.7 The Lotka-Volterra model for the predator-prey problem.

The Lotka-Volterra model can be used to study population dynamics problems as well as more general problems (such as chemical and nuclear reactions and game theory problems).

Let:

$$x_1 \equiv \text{number of prey (scaled, dimensionless)}$$

$$x_2 \equiv \text{number of predators (scaled, dimensionless)}$$

This system can be modeled as:

$$\dot{x}_1 = x_1 - x_1 x_2 \tag{2.47a}$$

$$\dot{x}_2 = x_1 x_2 - k x_2 \tag{2.47b}$$

In these equations the product $x_1 x_2$ can be considered to represent a "collision" between predator and prey. When collisions occur the prey decrease and the predators increase in number. Without collisions, the prey increase exponentially while the predators decrease exponentially with time constant k. This is actually a dynamics problem but it can be turned into an optimization problem.

Let us imagine the problem of a farmer raising crops (i.e. prey, x_1) which are damaged by insects (i.e. predators, x_2). The farmer wants to maximize profit, which depends on the crops produced at the end of the season, say 100 days. The farmer may elect to introduce insecticide to kill off the insects, but insecticide costs money.

Let the control variable be

$$u \equiv \text{rate of insecticide introduction}$$

Then,

$$\dot{x}_1 = x_1 - x_1 x_2 \tag{2.48a}$$

$$\dot{x}_2 = x_1 x_2 - k x_2 - l x_2 u, \quad (0 \leq l \leq 1) \tag{2.48b}$$

where l is considered the insecticide effectiveness. The farmer's cost function can then be stated as:

Minimize:

$$J = -x_{1f} + \int_0^{t_f} a_1 u \, dt \tag{2.49}$$

where $a_1 > 0$ (a constant), l, and t_f are given. (Here we use the negative sign with x_{1f} because the farmer wants to maximize crop output.)

To complete the setup of this optimization we need to specify I.C.s $x_1(0)$ and $x_2(0)$ as well as the final B.C.s:

$$\Psi_1 = t_f - 100 = 0 \tag{2.50}$$

A similar biological population dynamics problem is discussed in Stengel [1994].

2.5. Summary

The brachistochrone problem presented by Johann Bernoulli at the end of the seventeenth century was a new mathematical problem which required that a path (i.e., a function) be found to minimize a scalar function of that path. In the middle of the twentieth century a closely related problem to the path of quickest descent presented itself: the problem of finding the propellant-optimal trajectory for launching a satellite into orbit. Such problems that map functions (paths) into a scalar are called optimal control problems. They can be expressed generally in three forms: the Mayer problem (a function of the final time or final state), the Lagrange problem (a definite integral over time which includes the path function in the integrand), and the Bolza problem (which is a combination of the Mayer and the Lagrange problems). The three forms are equivalent; the text adopts the Problem of Bolza which most often appears in the literature.

The allowable (or admissible) class of functions for the control or for the trajectory can alter the solution obtained. What is admissible due to engineering or physical constraints may differ from what is mathematically admissible. For example, while the Dirac delta function is amenable to mathematical analysis in the field of generalized functions, it is physically impossible to mechanize as a control. To avoid unacceptable solutions (inadmissible functions) it may be possible to reformulate the problem, for example by putting bounds on the control or on its derivatives.

The Bolza problem is an ideal form for studying space trajectory optimization in which a control (such as the steering law for the thrust vector) is used to direct a launch vehicle into orbit using the least amount of propellant. While this text focuses on space trajectory optimization, it will include other examples of optimal control, such as the classical problem of Zermelo.

2.6. Exercises

1. Find the C space for $f(x_1, x_2) = x_1 + x_2^{3/2}$:

   1a. For $-\infty < x_1 < \infty,\ 0 \leq x_2 < \infty$
   1b. For $-\infty < x_1 < \infty,\ 0 < x_2 < \infty$

2. Let

$$\dot{x}_1 = \frac{1}{t}$$

$$\dot{x}_2 = x_1$$

   Assume $0 < t < \infty$. What class of functions are x_1 and x_2? (Answer in terms of P.C., C^0, C^1, etc.)

References

E.T. Bell, *Men* [sic] *of Mathematics* (Simon and Schuster, New York, 1965)

G.A. Bliss, *Lectures on the Calculus of Variations*. Phoenix Science Series (The University of Chicago Press, Chicago, 1968)

O. Bolza, *Lectures on the Calculus of Variations* (Dover, New York, 1961)

A.E. Bryson Jr., Y.C. Ho, *Applied Optimal Control* (Hemisphere Publishing, Washington, D.C., 1975)

D.G. Hull, *Optimal Control Theory for Applications* (Springer, New York, 2003)

W. Kaplan, *Operational Methods for Linear Systems* (Addison-Wesley, Reading, 1962)

C. Lanczos, *The Variational Principles of Mechanics*, 4th edn. (Dover, New York, 1986)

R.F. Stengel, *Optimal Control and Estimation* (Dover, New York, 1994)

J. Vagners, Optimization techniques, in *Handbook of Applied Mathematics*, ed. by C.E. Pearson, 2nd edn. (Van Nostrand Reinhold, New York, 1983), pp. 1140–1216